The Osculants of Plane Rational Quartic Curves.

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§ 1. *Cases of Degeneracy.*

We take the equations of the rational plane curve in the form

$$x_i = (\alpha_i t \rceil^n \equiv a_i t^n + b_i t^{n-1} + \dots + p_i, \qquad i = 1, 2, 3,$$

the arrowhead in $(\alpha_i t\rceil^n$ indicating the absence of binomial coefficients.

These equations in general represent a curve of order n. The order of the curve will, however, be less than n if:

1) There is a factor involving t common to the three $(\alpha_i t\rceil^n$;

2) It is possible by a substitution of the kind *

$$t' = \frac{(\beta t\rceil^m}{(\gamma t\rceil^m}$$

to reduce the equations to the form

$$x_i = (\alpha_i t\rceil^{\frac{n}{m}}.$$

In the latter case the original equations represent the curve of order $\frac{n}{m}$ repeated m times.

The case of a linear relation existing between the three x_i is included in this, since the equations

$$x_1 = (\alpha_1 t\rceil^n, \quad x_2 = (\alpha_2 t\rceil^n, \quad x_3 = k_1 (\alpha_1 t\rceil^n + k_2 (\alpha_2 t\rceil^n$$

are equivalent to

$$x_1 = t', \quad x_2 = 1, \quad x_3 = k_1 t + k_2,$$

where

$$t' = \frac{(\alpha_1 t\rceil^n}{(\alpha_2 t\rceil^n}.$$

* Where $\frac{n}{m}$ is a positive integer.

We postulate that, after all common factors are removed, the equations can not represent the product of distinct curves. It may be observed that a curve of genus zero is not necessarily rational. Thus the product of a conic and a cubic of genus one is a quintic of genus zero, but is not rational.*

For brevity we sometimes refer to the point of the curve given by $t = t_1$ as the point t_1.

We may make the equations homogeneous in the parameter by writing $\frac{t}{t'}$ for t. Then we may define the m-th osculant of the curve at the point $\frac{\tau}{\tau'}$ as the curve represented by the equations

$$x_i = (\tau D_t + \tau' D_{t'})^m (\alpha_i t)^n,$$

in which we may let $t' = \tau' = 1$, if desired. It is well known that osculants at the point τ touch the curve there, that all first osculants touch the stationary tangents of the curve, and that the $(n-1)$-st osculant is the tangent to the curve.

We shall see that osculants degenerate under certain conditions. As an example we may take the quartic

$$x_1 = t^4 + b t^3, \quad x_2 = t + e, \quad x_3 = t^2.$$

This curve is bicuspidal, having cusps at the points $(0, 1, 0)$ and $(1, 0, 0)$. If $b = 4e$, it has a third cusp at the point $(16e^4, e, -4e^2)$ given by $t = -2e$, the tangent at this point, $x_1 - 16e^3 x_2 = 0$, meeting the curve again at the point $(48e^4, 3e, 4e^2)$ given by $t = 2e$.

The first osculant of the curve at the point τ is

$$x_1 = (4\tau + b) t^3 + 3 b \tau t^2, \quad x_2 = 3t + \tau + 4e, \quad x_3 = 2t^2 + 2\tau t.$$

Let us first let $e = 1$, and form the osculant at the point $\tau = 2$. It is

$$x_1 = (8 + b) t^3 + 6 b t^2, \quad x_2 = 3(t + 2), \quad x_3 = 2t(t + 2).$$

The elimination† of t gives

$$4 x_1 x_2 (3 x_3 + 4 x_2) - 9 x_3^2 ((8 + b) x_3 + 4 b x_2) = 0,$$

which becomes, if $b = 4$,

$$4 (x_1 x_2 - 9 x_3^2)(4 x_2 + 3 x_3) = 0.$$

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* Clebsch-Lindemann: "Vorlesungen über Geometrie," Vol. I, p. 884.

† By "the elimination of t" we mean its elimination from two equations such as

$$\frac{x_1}{x_2} = \frac{(8 + b) t^3 + 6 b t^2}{3(t + 2)} \text{ and } \frac{x_1}{x_3} = \frac{(8 + b) t^3 + 6 b t^2}{2t(t + 2)}.$$

If we let, however, $b=4$ in the quartic and form the osculant at $\tau=2$, we have

$$x_1=12\,t^2(t+2),\quad x_2=3\,t+(2+4\,e),\quad x_3=2\,t\,(t+2),$$

or, on eliminating t,

$$x_1\,x_2\,(x_1+12\,x_3)-9\,x_3^2\,(x_1+2\,(2+4\,e)\,x_3)=0,$$

which becomes, if $e=1$,

$$(x_1\,x_2-9\,x_3^2)\,(x_1+12\,x_3)=0.$$

The lines $4\,x_2+3\,x_3=0$ and $x_1+12\,x_3=0$ meet the conic $x_1\,x_2-9\,x_3^2=0$ at the point (48, 3, —4) which is on the cusp tangent $x_1-16\,x_2=0$.

It seems, then, that this conic and any line on this point form an osculant cubic of the quartic at the point $t=2$. The first osculant at the point τ of

$$x_1=t^4+4\,t^3,\quad x_2=t+1,\quad x_3=t^2$$

being

$$x_1=4\,(\tau+1)\,t^3+12\,\tau\,t^2,\quad x_2=3\,t+\tau+4,\quad x_3=2\,t^2+2\,\tau\,t,$$

the first osculant of the latter curve at the point τ_1 is

$$\begin{aligned} x_1 &= 12\,(\tau\,\tau_1+\tau+\tau_1)\,t^2+24\,\tau\,\tau_1\,t,\\ x_2 &= 6\,t+3\,(\tau+\tau_1)+12,\\ x_3 &= 2t^2+4\,(\tau+\tau_1)\,t+2\,\tau\,\tau_1. \end{aligned}$$

If in the first of these curves we let $\tau=2$, we have $\tau+2=0$ and

$$x_1=12\,t^2,\quad x_2=3,\quad x_3=2\,t.$$

If in the second curve we let $\tau=\tau_1=2$, we have

$$x_1=96\,t\,(t+1),\quad x_2=6\,(t+4),\quad x_3=2\,(t^2+8\,t+4).$$

Hence the conic which forms part of the degenerate first osculant is not the second osculant.

It may be noted that osculants of degenerate curves are not, in general, degenerate. Thus the first osculant of

$$x_1=t^3,\quad x_2=3\,t^2,\quad x_3=3\,t$$

is

$$x_1=\tau\,t^2,\quad x_2=t^2+2\,\tau\,t,\quad x_3=2\,t+\tau,$$

and that of the repeated conic

$$x_1=t^4,\quad x_2=2\,t^2,\quad x_3=1$$

is

$$x_1=\tau t^3,\quad x_2=t^2+\tau t,\quad x_3=1.$$

The osculant cubics of

$$x_1 = t^4 + 4t^3, \quad x_2 = t + e, \quad x_3 = t^2$$

being

$$x_1 = 4(\tau + 1)t^3 + 12\tau t^2, \quad x_2 = 3t + \tau + 4e, \quad x_3 = 2t^2 + 2\tau t,$$

the locus of the nodes of these cubics is, by the formula of page 226, the sextic

$$x_1 = \tau^3(\tau + 4e)(\tau - 2)^2, \quad x_2 = (\tau + 1)(\tau - 2e)^2, \quad x_3 = \tau^2(\tau - 2)(\tau - 2e).$$

If $e = 1$, this becomes $(\tau - 2)^2 = 0$ and

$$x_1 = \tau^3(\tau + 4), \quad x_2 = \tau + 1, \quad x_3 = \tau^2,$$

which is exactly the form which the equation of the original quartic takes for this value of e. Hence we see that for a tricuspidal quartic part of the locus of nodes of osculant cubics is the quartic itself. It remains to find a meaning for the factor $(t - 2)^2$. If we write

$$\rho \equiv \frac{x_1}{x_3} = \frac{\tau(\tau + 4e)(\tau - 2)}{\tau - 2e}, \quad \sigma \equiv \frac{x_2}{x_3} = \frac{(\tau + 1)(\tau - 2e)}{\tau^2(\tau - 2)},$$

we have

$$\tau^3 + (4e - 2)\tau^2 - (8e + \rho)\tau + 2e\rho = 0$$

and

$$\sigma\tau^3 - (2\sigma + 1)\tau^2 + (2e - 1)\tau + 2e = 0,$$

or, if $\tau = \tau_1 + 2$,

$$\begin{aligned} \tau_1^3 + 4(e + 1)\tau_1^2 + (8e + 4 - \rho)\tau_1 + 2\rho(e - 1) &= 0, \\ \sigma\tau_1^3 + (4\sigma - 1)\tau_1^2 + (4\sigma + 2e - 5)\tau_1 + 6(e - 1) &= 0, \end{aligned}$$

the resultant of which equations gives the trilinear equation of the sextic.

The resultant of the equations*

$$at^3 + bt^2 + ct + d = 0 \quad \text{and} \quad a_1t^3 + b_1t^2 + c_1t + d_1 = 0$$

is

$$\begin{aligned} |ad_1|^3 - 2|ad_1|\,|ab_1|\,|cd_1| - |ad_1|\,|ac_1|\,|bd_1| + |ac_1|^2|cd_1| \\ + |bd_1|^2|ab_1| - |ab_1|\,|bc_1|\,|cd_1| = 0. \end{aligned}$$

In our case $e - 1$ is a factor of both d and d_1; therefore it is a factor of each of the determinants in which d occurs, and is a factor of the resultant. If we remove this factor, the remaining factor does not vanish identically if $e = 1$, but becomes

$$\frac{|cd_1|}{e - 1}\Big[|ac_1|^2 - |ab_1|\,|bc_1|\Big] = 0.$$

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* Salmon-Fiedler: "Alg. der Lin. Trans.," p. 93.

If $e = 1$,

$$\frac{|c\,d_1|}{e-1} \equiv \begin{vmatrix} 12-\rho, & 2\rho \\ 4\sigma - 3, & 6 \end{vmatrix} = -8(\sigma\rho - 9),$$

or this factor gives $x_1 x_2 - 9x_3^2 = 0$.

In like manner

$$|a\,c_1|^2 - |a\,b_1|\;|b\,c_1| = 0$$

gives

$$(x_1 x_2 - x_3^2)^2 - x_3^2(x_1 + 4x_3)(4x_2 + x_3) = 0,$$

which, as may easily be verified, is the original quartic.

Hence it appears that since the conic $x_1 x_2 - 9x_3^2 = 0$ and any line on the point $(48, 3, -4)$ form an osculant cubic at $t = 2$, any point of this conic is the node of an osculant cubic and the conic is a part of the nodal locus, corresponding to the factor $(t-2)^2$ in the parametric equation of that locus.

We meet with degeneracies of the kind now being considered when we form the line equations of cuspidal curves. Thus the cubic

$$x_1 = t^3, \quad x_2 = 1, \quad x_3 = t^2 + ct$$

becomes cuspidal if $c = 0$. The equation of its tangent is

$$(2t + c)x_1 + t^3(t + 2c)x_2 - 3t^2 x_3 = 0,$$

and its line equations are

$$\xi_1 = 2t + c, \quad \xi_2 = t^3(t + 2c), \quad \xi_3 = -3t^2.$$

If $c = 0$, this becomes $t = 0$ and

$$\xi_1 = 2, \quad \xi_2 = t^3, \quad \xi_3 = -3t,$$

or

$$27\,\xi_1^2\,\xi_2 + 4\,\xi_3^3 = 0.$$

The elimination of t from the equations of the quartic gives

$$(9\,\xi_1\,\xi_2 - c\,\xi_3^2)^2 - 4\,\xi_3^2(3c\,\xi_1 - \xi_3)(3\,\xi_2 - c^2\,\xi_3) = 0.$$

The points of inflexion are

$$\xi_1 = 0, \quad \xi_2 = 0 \text{ and } c^3\xi_1 - \xi_2 = 0,$$

and the node is

$$c^3\xi_1 + \xi_2 - c^2\xi_3 = 0.$$

If $c = 0$, the quartic becomes

$$3\,\xi_2(27\,\xi_1^2\,\xi_2 + 4\,\xi_3^3) = 0.$$

Two points of inflexion and the node are $\xi_2 = 0$, which point is a part of the quartic, and is also the cusp of the cuspidal cubic which forms the other part.

The pencil of conics on the three stationary tangents and the tangent at $t=1$ of the quartic is

$$(3\,\xi_2-c^2\xi_3)\,(3\,\xi_1+(2+c)\,\xi_3)+\tau\,[(3\,c\,\xi_1-\xi_3)\,(3\,\xi_2+(1+2\,c)\,\xi_3)]=0,$$

and the eighth common line is given by

$$2\,t+c\,(t+1)-\tau\,(t+1+2\,c)=0.$$

If $c=0$, this becomes

$$2\,t-\tau\,(t+1)=0.$$

If $t=0$, $\tau=0$ and for this value of τ, the conic of the pencil is

$$3\,\xi_2\,(3\,\xi_1+2\,\xi_3)=0\,;$$

i. e., the point $\xi_2=0$ and the meet of the stationary tangent of the cubic and the tangent at $t=1$.

In this case we find that, when we give the parameter an evident geometrical meaning, one part of the degenerate curve corresponds to a particular value of the parameter.

We now give an example of a conic degenerating into a repeated line. The osculant conic at the point τ of the cubic

$$x_1=3\,t^2, \quad x_2=3\,t, \quad x_3=t^3-1$$

is

$$x_1=t^2+2\,\tau\,t, \quad x_2=2\,t+\tau, \quad x_3=\tau t^2-1.$$

If $\tau=1$, this becomes

$$x_1=t^2+2\,t, \quad x_2=2\,t+1, \quad x_3=t^2-1\equiv t^2+2\,t-(2\,t+1),$$

or, if $t'=\dfrac{t^2+2\,t}{2\,t+1}$,

$$x_1=t', \quad x_2=1, \quad x_3=t'-1.$$

Hence the osculant at this point, which is a point of inflexion of the cubic, is the stationary tangent $x_1-x_2-x_3=0$, repeated.

The tangent to the cubic at the point τ_1 meets the stationary tangent, $x_1-x_2-x_3=0$, at a point P. The other tangents to the curve from P are the stationary tangent counted twice and an additional line touching the curve at a point τ_2. It may easily be shown that τ_1 and τ_2 form a pair of the involution defined by $2\,\tau_1\tau_2+\tau_1+\tau_2+2=0$, and that τ_1 and τ_2 give the same point on the repeated line. The double points of the involution are

given by $\tau^2 + \tau + 1 = 0$, and ω and ω^2 give* on the line the points where the two other stationary tangents meet it. We shall see later† that the line equation of this osculant takes the form of the product of the equations of these two points.

The result of eliminating t from the equations of an osculant is

$$(\tau^2 x_1 - x_2 - \tau x_3)^2 - 4(\tau x_1 - \tau^2 x_2 - x_3)(x_1 - \tau x_2 - \tau^2 x_3) = 0,$$

which becomes, if $\tau = 1$,

$$-3(x_1 - x_2 - x_3)^2 = 0.$$

The four points of intersection, other than the point of contact, of the cubic and the osculant conic are given by

$$\tau^2 t^4 - 4t^3 + 6\tau t^2 - 4\tau^2 t + 1 = 0,$$

for which quartic the invariant $g_2 = 0$, its roots forming a self-apolar set.

Lüroth‡ has shown how to reduce the equation of a repeated curve. Brill§ states certain conditions under which a quartic degenerates, and gives an interesting note in regard to degeneracies due to the occurrence of a common factor.

§ 2. *The Fundamental Involution.*

The equations of the curve being $x_i = (\alpha_i t)^n$, we can find $n-2$ linearly independent forms of the n-th order apolar to the three $(\alpha_i t)^n$.

If $(\beta_0 t)^n, \ldots, (\beta_{n-3} t)^n$ are $n-2$ such forms, the equation

$$\lambda_0 (\beta_0 t)^n + \ldots + \lambda_{n-3} (\beta_{n-3} t)^n = 0$$

defines the fundamental involution of the curve.‖ If we assume $n-3$ roots of this equation, the remaining three are fixed. There is no fundamental involution for the conic unless it is degenerate.

For the cubic it is well known that the roots of $(\beta_0 t)^3 = 0$ are the parameters of the points of inflexion.

The condition that

$$t^4 - s_1 t^3 + s_2 t^2 - s_3 t + s_4 = 0$$

be apolar to

$$a_i t^4 + b_i t^3 + c_i t^2 + d_i t + e_i = 0$$

is

$$a_i s_4 + b_i \frac{s_3}{4} + c_i \frac{s_2}{6} + d_i \frac{s_1}{4} + e_i = 0,$$

which is the result of completely polarizing the latter equation.

* ω and ω^2 being cube roots of unity.

† Page 228.

‡ *Math. Annalen*, Vol. IX, p. 163.

§ *Math. Annalen*, Vol. XII, p. 101.

‖ Stahl: *Crelle*, Vol. CI, p. 300. Meyer: "Apolarität und Rationale Curven."

Hence we can write the fundamental involution of the quartic in the form:

$$\begin{vmatrix} t^4 + s_4 & a_1 s_4 + e_1 & a_2 s_4 + e_2 & a_3 s_4 + e_3 \\ -t^3 & \frac{d_1}{4} & \frac{d_2}{4} & \frac{d_3}{4} \\ t^2 & \frac{c_1}{6} & \frac{c_2}{6} & \frac{c_3}{6} \\ -t & \frac{b_1}{4} & \frac{b_2}{4} & \frac{b_3}{4} \end{vmatrix} = 0.$$

This form of the involution might have to be modified for special cases, but it is obvious that the fundamental involution of any rational curve may be found by the method here used.

We shall call the fundamental involution the I_n.

If a curve is given by $x_i = (\alpha_i t)^n$, using for the moment binomial coefficients, the three conditions that the n parameters $(t_1, t_2, \ldots, t_n)$ be a set of the I_n are

$$(\alpha_i t_1)(\alpha_i t_2) \ldots (\alpha_i t_n) = 0.$$

The first osculant at the point t_1 is $x_i = (\alpha_i t_1)(\alpha_i t)^{n-1}$, and the conditions that $(t_2, t_3, \ldots, t_n)$ be a set of the I_{n-1} of this osculant are again

$$(\alpha_i t_1)(\alpha_i t_2) \ldots (\alpha_i t_n) = 0.$$

Hence any set of the I_{n-1} on the osculant at t_1 form, with t_1, a set of the I_n on the original curve.

Thus for the quartic, as Stahl has shown, t_1 and the parameters of the points of inflexion of the osculant cubic at t_1 form a set of the I_4. In general, if we select the $n-3$ parameters $(t_1, t_2, \ldots, t_{n-3})$, the other three members of the set of the I_n which these determine are the parameters of the points of inflexion of the common osculant cubic

$$x_i = (\alpha_i t)^3 (\alpha_i t_1)(\alpha_i t_2) \ldots (\alpha_i t_{n-3}).$$

By a common osculant we mean a curve such as $x_i = (\alpha_i t_1)(\alpha_i t_2)(\alpha_i t)$ which is evidently both the tangent at t_2 of $x_i = (\alpha_i t_1)(\alpha_i t)^2$ and the tangent at t_1 of $x_i = (\alpha_i t_2)(\alpha_i t)^2$.

If we completely polarize

$$x_1 = 3t^2, \quad x_2 = 3t, \quad x_3 = t^3 - 1,$$

we have

$$x_1 = s_2, \quad x_2 = s_1, \quad x_3 = s_3 - 1.$$

The points of inflexion of this curve are given by $t^3 - 1 = 0$, and the stationary tangents are

$$x_1 - x_2 - x_3 = 0, \quad x_1 - \omega x_2 - \omega^2 x_3 = 0, \quad x_1 - \omega^2 x_2 - \omega x_3 = 0.$$

If in the above polarized equations we put $s_1 = s_2 = 0$, $s_3 = 1$, we have $x_1 = x_2 = x_3 = 0$. We may partially polarize the equations, writing

$$x_1 = \sigma_2 + \sigma_1 t, \quad x_2 = \sigma_1 + t, \quad x_3 = \sigma_2 t - 1.$$

In general, the two points given by $t^2 - \sigma_1 t + \sigma_2 = 0$ will in this way determine a line; but if the roots of this quadratic are the parameters of two points of inflexion, the line degenerates into a point. Thus if

$$t^2 - \sigma_1 t + \sigma_2 \equiv (t - \omega)(t - \omega^2) \equiv t^2 + t + 1,$$

we have

$$\begin{aligned} x_1 &= 1 - t &&\equiv -1 \,.\, (t - 1), \\ x_2 &= -1 + t &&\equiv 1 \,.\, (t - 1), \\ x_3 &= t - 1 &&\equiv 1 \,.\, (t - 1). \end{aligned}$$

The point $(-1, 1, 1)$ is the meet of the stationary tangents

$$x_1 - \omega x_2 - \omega^2 x_3 = 0 \quad \text{and} \quad x_1 - \omega^2 x_2 - \omega x_3 = 0.$$

If we partially polarize the equation of the quartic

$$x_i = a_i t^4 + 4 b_i t^3 + 6 c_i t^2 + 4 d_i t + e_i,$$

writing

$$x_i = a_i s_3 t + b_i (s_3 + s_2 t) + c_i (s_2 + s_1 t) + d_i (s_1 + t) + e_i,$$

or

$$x_i = (a_i s_3 + b_i s_2 + c_i s_1 + d_i)\left(t + \frac{b_i s_3 + c_i s_2 + d_i s_1 + e_i}{a_i s_3 + b_i s_2 + c_i s_1 + d_i}\right),$$

in general the three parameters (t_1, t_2, t_3) given by $t^3 - s_1 t^2 + s_2 t - s_3 = 0$ determine a line. If, however, they belong to a set of the I_4, of which t_4 is the fourth member, we know that $x_i = 0$ if $t = t_4$. Hence the above expressions for x_i are divisible by $t - t_4$ and the line degenerates into a point. The points of inflexion of the osculant cubic at t_3 are given by t_1, t_2 and t_4, and if we partially polarize its equations and substitute $t_1 + t_2$ for σ_1, and $t_1 t_2$ for σ_2, we evidently have the same point as the above. But we have seen that the point obtained by the latter method is the meet of the stationary tangents at t_1 and t_2 of the osculant cubic at t_3.

Obviously the stationary tangents at t_2 and t_3 of the osculant cubic at t_1, and the stationary tangents at t_3 and t_1 of the osculant cubic at t_2, are on this

point. The locus of this point is, then, the conic which Stahl* calls K; *i. e.*, the locus of meets of stationary tangents of osculant cubics. For a set of the I_4 of which α is a member we have the three equations

$$-\alpha = \frac{b_i s_3 + c_i s_2 + d_i s_1 + e_i}{a_i s_3 + b_i s_2 + c_i s_1 + d_i}.$$

From these equations we can express s_1, s_2 and s_3 in terms of α; and, substituting these values in

$$x_i = a_i s_3 + b_i s_2 + c_i s_1 + d_i,$$

we have the equations of K with α as the parameter in essentially the same form as given by Stahl.

Stahl has shown that if t_1, t_2, t_3 and t_4 form a set of the I_4, any two of the osculant cubics at these points have a common stationary tangent. There are thus in all six lines arranged as in the following diagram (Fig. 1).

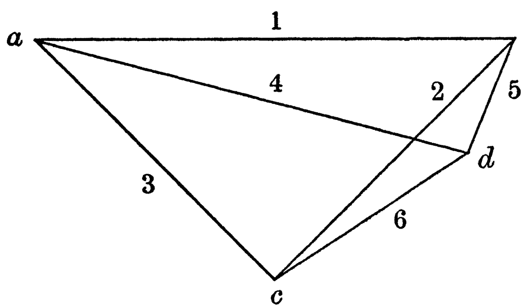

FIG. 1.

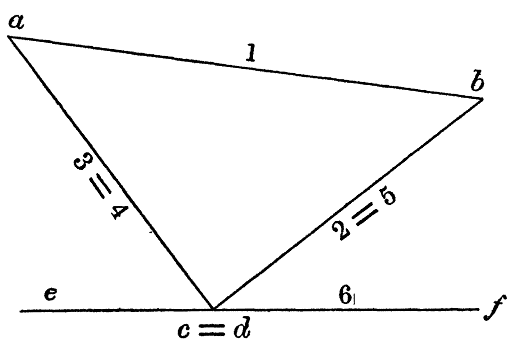

FIG. 2.

The four points of K determined by a set of the I_4 are a, b, c and d. If the lines 1, 2 and 3 are the stationary tangents of the osculant at t_4, d is the point of K given by $\alpha = t_4$.

§ 3. *The Covariant Curves g_2 and g_3.*

If we assume that the parameters of two of the points of inflexion of a plane rational quartic are 0 and ∞, we may, in general, write its equations

$$x_1 = a_1 t^4 + b_1 t^3, \quad x_2 = d_2 t + e_2, \quad x_3 = b_3 t^3 + c_3 t^2 + d_3 t.$$

We shall call the curve Q.

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* *Crelle*, Vol. CI, p. 306.

If $b_3 = 0$ or $d_3 = 0$, the curve is cuspidal; if $b_1 = 0$ or $d_2 = 0$, it has a point of undulation. The first osculant at the point τ may be written

$$x_1 = (4\,a_1\tau + b_1)\,t^3 + 3\,b_1\tau\,t^2, \quad x_2 = 3\,d_2\,t + d_2\tau + 4\,e_2,$$
$$x_3 = b_3\,t^3 + (3\,b_3\tau + 2\,c_3)\,t^2 + (2\,c_3\tau + 3\,d_3)\,t + d_3\tau.$$

If $\tau = 0$, this becomes

$$x_1 = b_1 t^3, \quad x_2 = 3\,d_2\,t + 4\,e_2, \quad x_3 = b_3 t^3 + 2\,c_3 t^2 + 3\,d_3 t,$$

whence it is evident that the osculant cubic at a point of inflexion has itself a point of inflexion there, and that the osculant cubic at a point of undulation is the undulation tangent counted three times (since for each value of $\frac{x_2}{x_3}$ there are three values of t).

If $d_3 = 0$, all the cubic osculants touch the cusp tangent, $x_1 = 0$, at the cusp. The inflexion tangents ($x_1 = 0$ and $x_2 = 0$) meet Q again at the points given by $a_1 t + b_1 = 0$ and $d_2 t + e_2 = 0$; the osculant cubic at the point $\tau = \frac{-4\,e_2}{d_2}$ touches $x_1 = 0$ at the meet of these inflexion tangents, while the osculant at $\tau = -\frac{b_1}{4\,a_1}$ touches $x_2 = 0$ at the same point.

The equations of the osculant in lines are

$$\begin{aligned}
\xi_1 = {} & 6\,b_3 d_2 t^3 + 3\,(b_3(d_2\tau + 4\,e_2) + d_2(3\,b_3\tau + 2\,c_3))\,t^2 \\
& + 2\,(d_2\tau + 4\,e_2)(3\,b_3\tau + 2\,c_3)\,t + 2\,(c_3 d_2\tau^2 + 4\,c_3 e_2\tau + 6\,d_3 e_2), \\
\xi_2 = {} & t\,[2\,(6\,a_1 b_3\tau^2 + 4\,a_1 c_3\tau + b_1 c_3)\,t^3 + 2\,(4\,a_1\tau + b_1)(2\,c_3\tau + 3\,d_3)\,t^2 \\
& + 3\,\tau\,(d_3(4\,a_1\tau + b_1) + b_1(2\,c_3\tau + 3\,d_3))\,t + 6\,b_1 d_3\tau^3], \\
\xi_3 = {} & -3\,t\,[2\,d_2(4\,a_1\tau + b_1)\,t^2 + ((4\,a_1\tau + b_1)(d_2\tau + 4\,e_2) + 3\,b_1 d_2\tau)\,t \\
& + 2\,b_1\tau\,(d_2\tau + 4\,e_2)].
\end{aligned}$$

If $\tau = t$ in these equations, we have the line equations of Q; viz.,

$$\begin{aligned}
\xi_1 &= 12\,[2\,b_3 d_2 t^3 + (3\,b_3 e_2 + c_3 d_2)\,t^2 + 2\,c_3 e_2 t + d_3 e_2], \\
\xi_2 &= 12\,t^3\,[a_1 b_3 t^3 + 2\,a_1 c_3 t^2 + (3\,a_1 d_3 + b_1 c_3)\,t + 2\,b_1 d_3], \\
\xi_3 &= -12\,t^2\,[3\,a_1 d_2 t^2 + 2\,(2\,a_1 e_2 + b_1 d_2)\,t + 3\,b_1 e_2].
\end{aligned}$$

The four points in which Q meets the line $(x\,\xi) = 0$* are given by

$$a_1\xi_1 t^4 + (b_1\xi_1 + b_3\xi_3)\,t^3 + c_3\xi_2 t^2 + (d_2\xi_2 + d_3\xi_3)\,t + e_2\xi_2 = 0.$$

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* $(x\,\xi) \equiv x_1\xi_1 + x_2\xi_2 + x_3\xi_3$.

The roots of this equation form a self-apolar set if the line touches the conic

$$g_2 \equiv (c_3^2 - 3\,b_3 d_3)\,\xi_3^2 - 3\,b_3 d_2 \xi_2 \xi_3 - 3\,b_1 d_3 \xi_1 \xi_3 + 3\,(4\,a_1 e_2 - b_1 d_2)\,\xi_1 \xi_2 = 0,$$

or, in points,

$$3(b_3 d_2 x_1 + b_1 d_3 x_2 + (4\,a_1 e_2 - b_1 d_2)\,x_3)^2 + 4(c_3^2(4\,a_1 e_2 - b_1 d_2) - 12\,a_1 b_3 d_3 e_2)\,x_1 x_2 = 0.$$

If $d_3 = 0$, Q has a cusp at $(0, 1, 0)$; g_2 then touches the cusp tangent at the cusp. The points where g_2 meets Q are given by

$$\begin{aligned} 3\,(a_1 b_3 d_2 t^4 + 4\,a_1 b_3 e_2 t^3 - 4\,a_1 d_3 e_2 t - b_1 d_3 e_2)^2 + 36\,a_1 b_3 d_3 e_2 (4\,a_1 e_2 - b_1 d_2)\,t^4 \\ + 6\,c_3 (4\,a_1 e_2 - b_1 d_2)\,(a_1 b_3 d_2 t^4 + 4\,a_1 b_3 e_2 t^3 + 4\,a_1 d_3 e_2 t + b_1 d_3 e_2)\,t^2 \\ + c_3^2 (4\,a_1 e_2 - b_1 d_2)\,(4\,a_1 t + b_1)\,(d_2 t + 4\,e_2)\,t^3 = 0. \end{aligned}$$

The discriminant of g_2 is

$$\frac{9}{4}(4\,a_1 e_2 - b_1 d_2)\,(12\,a_1 b_3 d_3 e_2 - c_3^2 (4\,a_1 e_2 - b_1 d_2)).$$

If $(4\,a_1 e_2 - b_1 d_2) = 0$, g_2 becomes

$$\xi_3 ((c_3^2 - 3\,b_3 d_3)\,\xi_3 - 3\,b_3 d_2 \xi_2 - 3\,b_1 d_3 \xi_1) = 0$$

and three stationary tangents of Q meet at each of the points given by this equation. Four stationary tangents can not be on the same point, since then there would be eight tangents to Q from this point.

If $d_3 = 0$, g_2 meets Q at the points

$$\begin{aligned} 3\,a_1^2 b_3^2 (d_2 t + 4\,e_2)^2 t^6 + 6\,a_1 b_3 c_3 (4\,a_1 e_2 - b_1 d_2)\,(d_2 t + 4\,e_2)\,t^5 \\ + c_3^2 (4\,a_1 e_2 - b_1 d_2)\,(4\,a_1 t + b_1)\,(d_2 t + 4\,e_2)\,t^3 = 0; \end{aligned}$$

i. e., g_2 touches the cusp tangent at the cusp and one of the five other points of intersection is given by $d_2 t + 4\,e_2 = 0$.

If also $b_3 = 0$, and $4\,a_1 e_2 - b_1 d_2 = 0$, the quartic is tricuspidal: g_2 is then $c_3^2 \xi_3^2 = 0$, and the equation giving its intersections with Q vanishes identically.

Obviously g_2 touches the stationary tangents of Q: it touches $x_1 = 0$ and $x_2 = 0$ at the points $(0,\ 4\,a_1 e_2 - b_1 d_2,\ -b_1 d_3)$ and $(4\,a_1 e_2 - b_1 d_2,\ 0,\ -b_3 d_2)$ respectively.

Knowing a stationary tangent of Q, we can put its points in one-to-one correspondence with the lines of g_2. Thus the line $x_2 = 0$ touches the osculant at the point given by $t = \infty$, viz.:

$$x_1 = 4\,a_1 \tau + b_1, \quad x_2 = 0, \quad x_3 = b_3.$$

The equation of this point is $(4\,a_1 \tau + b_1)\,\xi_1 + b_3 \xi_3 = 0$, the correspondence between ξ_1/ξ_3 and τ being one-to-one.

From this point there is one tangent, other than $x_2 = 0$, to g_2. If in the line equation of g_2 we substitute $\xi_1 = -\dfrac{b_3 \xi_3}{4 a_1 \tau + b_1}$, we obtain parametric equations of the conic in the form

$$\begin{aligned}
\xi_1 &= -12 a_1 b_3^2 (d_2 \tau + e_2),\\
\xi_2 &= (c_3^2 (4 a_1 \tau + b_1) - 12 a_1 b_3 d_3 \tau)(4 a_1 \tau + b_1),\\
\xi_3 &= 12 a_1 b_3 (d_2 \tau + e_2)(4 a_1 \tau + b_1).
\end{aligned}$$

If $b_3 = 0$, all the first osculants touch $x_2 = 0$ at the cusp and these equations can not be used. The osculant at $\tau = -\dfrac{e_2}{d_2}$ is

$$\begin{gathered}
x_1 = -(4 a_1 e_2 - b_1 d_2) t^3 - 3 b_1 e_2 t^2, \quad x_2 = 3 d_2 (d_2 t + e_2),\\
x_3 = b_3 d_2 t^3 - (3 b_3 e_2 - 2 c_3 d_2) t^2 - (2 c_3 e_2 - 3 d_2 d_3) t - d_3 e_2.
\end{gathered}$$

If $t = \infty$,

$$x_1 = -(4 a_1 e_2 - b_1 d_2), \quad x_2 = 0, \quad x_3 = b_3 d_2.$$

Hence the osculant cubic of a quartic, at the point at which a tangent at a point of inflexion meets it again, touches that tangent at the point at which g_2 touches it. Since the above correspondence between τ and ξ_1/ξ_3 is one-to-one, it is evident that, in general, there is no pair of points on Q such that the osculant cubics at these points coincide.

There are eight tangents common to g_2 and an osculant cubic. Of these six are the stationary tangents of Q. The two variable tangents are given, on the osculant, by

$$\begin{aligned}
(d_2 \tau + e_2) \rho_\tau t^2 + 2\,[3 a_1 b_3 \tau^3 (d_2 \tau + 4 e_2) + c_3 (4 a_1 \tau + b_1)(d_2 \tau + 4 e_2)\tau&\\
+ 3 d_3 e_2 (4 a_1 \tau + b_1)]\, t - \tau (a_1 \tau + b_1) \sigma_\tau &= 0,
\end{aligned}$$

where

$$\begin{aligned}
\rho_\tau &= 2 (6 a_1 b_3 \tau^2 + 4 a_1 c_3 \tau + b_1 c_3),\\
\sigma_\tau &= -2 (c_3 d_2 \tau^2 + 4 c_3 e_2 \tau + 6 d_3 e_2).
\end{aligned}$$

Two of the eight tangents will coincide if a variable tangent coincides with one of the stationary tangents, or if the two variable tangents coincide. Thus $t = 0$ gives $x_1 = 0$. One of the variable tangents is given by $t = 0$ if:

(a) $\tau = 0$, *i. e.*, $x_1 = 0$ is a stationary tangent of the osculant;

(b) $a_1 \tau + b_1 = 0$, when the osculant touches g_2 where $x_1 = 0$ touches it;

(c) $\sigma_\tau = 0$, in which case, as is evident from its line equations, the osculant is cuspidal and has only six proper tangents in common with g_2.

The two variable tangents coincide if

$$\tau(a_1\tau+b_1)(d_2\tau+e_2)\rho_\tau\sigma_\tau+[3a_1b_3\tau^3(d_2\tau+4e_2)$$
$$+c_3(4a_1\tau+b_1)(d_2\tau+4e_2)\tau+3d_3e_2(4a_1\tau+b_1)]^2=0.$$

This equation is of the eighth order in τ and on expansion proves to be identical with that giving the intersections of Q with g_2.

The osculant at such a point has a node at the point of contact, the nodal tangents being the tangent to Q and the tangent to g_2. It follows that the node of an osculant cubic of a tricuspidal quartic is at the point of contact and that one of the nodal tangents passes through the meet of the cusp tangents.

The points of contact of the two tangents to the cubic

$$x_i=a_it^3+b_it^2+c_it+d_i$$

from the point τ on it are given by the equation

$$[|abc|\tau+|abd|]t^2+2[|abd|\tau+|acd|]t+|acd|\tau+|bcd|=0.$$

If we apply this formula to the cubic osculant of Q at τ, *we find that the two tangents to the osculant from the point of contact coincide with the two variable common tangents of g_2 and the osculant.*

The equations of the conic K are

$$x_1=-3b_1^2\sigma_\alpha,\qquad x_2=3d_2^2\rho_\alpha,$$
$$x_3=3(d_2d_3\rho_\alpha-b_1b_3\sigma_\alpha)+144a_1b_3d_3e_2\alpha-4c_3^2(4a_1\alpha+b_1)(d_2\alpha+4e_2).$$

If $b_1=0$, $x_1=0$ is an undulation tangent of Q, and K is degenerate. In fact, it is evident that an undulation tangent of Q is a stationary tangent of each cubic osculant, and that, therefore, every point on it is a point of K.

If $d_3=0$, K becomes

$$x_1=6b_1^2c_3(d_2\alpha+4e_2)\alpha,\qquad x_2=3d_2^2\rho_\alpha,$$
$$x_3=2(d_2\alpha+4e_2)(3b_1b_3c_3\alpha-2c_3^2(4a_1\alpha+b_1)).$$

K is on the cusp, but does not touch the cusp tangent.

In general, $x_1=0$ meets K when $\rho_\alpha=0$. The equation of the two points given by these values of α is

$$3d_2^2\xi_2^2+2(3d_2d_3-4c_3e_2)\xi_2\xi_3+3d_3^2\xi_3^2=0.$$

The locus of lines such that the parameters of the points in which they cut Q form harmonic sets is

$$g_3 \equiv \begin{vmatrix} a_1\xi_1, & \frac{b_1\xi_1 + b_3\xi_3}{4}, & \frac{c_3}{6} \\ \frac{b_1\xi_1 + b_3\xi_3}{4}, & \frac{c_3}{6}, & \frac{d_2\xi_2 + d_3\xi_3}{4} \\ \frac{c_3}{6}, & \frac{d_2\xi_2 + d_3\xi_3}{4}, & e_2\xi_2 \end{vmatrix} = 0,$$

or

$$c_3\xi_3(72\,a_1e_2\xi_1\xi_2 + 9\,(b_1\xi_1 + b_3\xi_3)(d_2\xi_2 + d_3\xi_3) - 2\,c_3^2\xi_3^2) \\ - 27\,a_1\xi_1(d_2\xi_2 + d_3\xi_3)^2 - 27\,e_2\xi_2(b_1\xi_1 + b_3\xi_3)^2 = 0.$$

Evidently g_3 touches Q at the points of inflexion.

If the roots of the equation $a_0t^3 + 3\,a_1t^2 + 3\,a_2t + a_3 = 0$ are equal,

$$a_0t + a_1 = 0, \quad a_1t + a_2 = 0 \text{ and } a_2t + a_3 = 0.$$

A cubic osculant meets the line $(x\xi) = 0$ when

$$((4\,a_1\tau + b_1)\xi_1 + b_3\xi_3)t^3 + (3\,b_1\tau\xi_1 + (3\,b_3\tau + 2\,c_3)\xi_3)t^2 \\ + (3\,d_2\xi_2 + (2\,c_3\tau + 3\,d_3)\xi_3)t + (d_2\tau + 4e_2)\xi_2 + d_3\tau\xi_3 = 0.$$

If the line is a stationary tangent of the osculant, we have from the first of the above conditions

$$3\,((4\,a_1\tau + b_1)\xi_1 + b_3\xi_3)t + 3\,b_1\tau\xi_1 + (3\,b_3\tau + 2\,c_3)\xi_3 = 0,$$

or

$$12\,a_1\xi_1\tau t + 3\,(b_1\xi_1 + b_3\xi_3)(\tau + t) + 2\,c_3\xi_3 = 0.$$

In like manner the other conditions give

$$3\,(b_1\xi_1 + b_3\xi_3)\tau t + 2\,c_3\xi_3(\tau + t) + 3\,(d_2\xi_2 + d_3\xi_3) = 0$$

and

$$2\,c_3\xi_3\tau t + 3\,(d_2\xi_2 + d_3\xi_3)(\tau + t) + 12\,e_2\xi_2 = 0.$$

Eliminating τt and $\tau + t$, we find that the envelope of the stationary tangents of osculant cubics is g_3.*

In general, a set of the I_4 of Q determines four points on K. Let us consider what happens if the set is one to which the parameter of a point of inflexion of Q (say $t = 0$) belongs. One of the stationary tangents of the cubic osculant at this point is $x_1 = 0$, given by $t = 0$; hence 0 is to be counted twice as a member of this set. We may call the two other members α and β. The osculant at $t = 0$

* Morley: *Trans. Am. Math. Soc.*, Vol. VIII, p. 17. Stahl: *Crelle*, Vol. CI, p. 302.

has three distinct stationary tangents given by $t=0$, $t=\alpha$ and $t=\beta$; the osculant at $t=\alpha$ and the osculant at $t=\beta$ have each two stationary tangents given by $t=0$, and one given by $t=\beta$ and $t=\alpha$ respectively; *i. e.*, these latter osculants are cuspidal. The points c and d of Fig. 1 (page 216) coincide as in Fig. 2. The stationary tangents of the osculant at $t=0$ are ab, bc and ca; bc and ca are cuspidal tangents of the osculants at $t=\alpha$ and $t=\beta$ respectively, while these two osculants have a common stationary tangent, ef. The line $x_1=0$ is not a proper tangent of the cuspidal osculants, but merely passes through the cusps.

In general, from a point of K there are three distinct tangents to g_3. At each of the points a and b two of these tangents to g_3 coincide. Hence a and b are on g_3, bc and ca being tangents of g_3.

Since, in general, g_3 is of the sixth order, it intersects K in twelve points which are thus accounted for.

Of the thirty-six intersections of the two sextics, g_3 and the six stationary tangents of Q, twelve are on the conic K; there is, therefore, a curve of the fourth order touching g_3 at the six points of inflexion of Q and passing through the other twelve intersections.

The line ef, being the limiting position of the chord cd of K, touches K. *In this way we can account for the six common tangents of g_3 and K.*

§ 4. *The Nodal Locus of Osculant Cubics.*

Stahl has shown that the envelope of the lines joining the points of inflexion of osculant cubics of Q is a conic. The locus of the nodes of these cubics is evidently a rational curve, the equations of which we will now find.

In general, the equations of a rational cubic may by a change of reference triangle be put in the form

$$x_1 = A_1 t^3 + B_1 t^2, \quad x_2 = C_2 t + D_2, \quad x_3 = A_3 t^3 + B_3 t^2 + C_3 t + D_3.$$

The line $(x\xi)=0$ meets this curve at the points

$$(\xi_1 A_1 + \xi_3 A_3) t^3 + (\xi_1 B_1 + \xi_3 B_3) t^2 + (\xi_2 C_2 + \xi_3 C_3) t + (\xi_2 D_2 + \xi_3 D_3) = 0.$$

The quadratic $m_0 t^2 + m_1 t + m_2 = 0$ is a factor of this if

$$\frac{m_0(\xi_2 C_2 + \xi_3 C_3) - m_2(\xi_1 A_1 + \xi_3 A_3)}{m_0(\xi_1 B_1 + \xi_3 B_3) - m_1(\xi_1 A_1 + \xi_3 A_3)} = \frac{m_1}{m_0},$$

and

$$\frac{m_0(\xi_2 D_2 + \xi_3 D_3)}{m_0(\xi_1 B_1 + \xi_3 B_3) - m_1(\xi_1 A_1 + \xi_3 A_3)} = \frac{m_2}{m_0}.$$

In general, these two equations determine the ratios $\xi_1 : \xi_2 : \xi_3$, and thus a definite line. If, however, $m_0 t^2 + m_1 t + m_2 = 0$ gives the parameters of the double point of the cubic, the two equations are equivalent and give merely the equation of the double point. For this cubic the parameters of the points of inflexion are given by

$$C_2 R t^3 + 3 D_2 R t^2 - 3 A_1 S t - B_1 S = 0,$$

and those of the double point by

$$(D_2^2 R + A_1 C_2 S) R t^2 - (A_1 D_2 - B_1 C_2) R S t + (B_1 D_2 R + A_1^2 S) S = 0,$$

where

$$R = A_1 B_3 - A_3 B_1 \quad \text{and} \quad S = C_2 D_3 - C_3 D_2.$$

The cubic is cuspidal if

$$R S [(A_1 D_2 - B_1 C_2)^2 R S - 4 (D_2^2 R + A_1 C_2 S)(B_1 D_2 R + A_1^2 S)] = 0.$$

Using the second of the conditions given above, we find, after removal of the common factor $R D_2$, for the coordinates of the double point:

$$\begin{aligned} x_1 &= -(B_1 D_2 R + A_1^2 S)^2 S, \qquad x_2 = (D_2^2 R + A_1 C_2 S)^2 R, \\ x_3 &= D_2^3 D_3 R^3 - D_2 (B_1 B_3 D_2 - 2 A_1 C_2 D_3) R^2 S \\ &\quad + A_1 (A_1 C_2 C_3 - 2 A_3 B_1 D_2) R S^2 - A_1^3 A_3 S^3 - (A_1 D_2 + B_1 C_2) R^2 S^2. \end{aligned}$$

If $D_3 = 0$, we have

$$\begin{aligned} x_1 &= C_3 D_2^3 (B_1 R - A_1^2 C_3)^2, \qquad x_2 = D_2^2 (D_2 R - A_1 C_2 C_3)^2 R, \\ x_3 &= D_2^2 [(B_1 B_3 C_3 D_2 - A_1 C_3^2 D_2 - B_1 C_2 C_3^2) R^2 \\ &\qquad + A_1 C_3^2 (A_1 C_2 C_3 - 2 A_3 B_1 D_2) R + A_1^3 A_3 C_3^3 D_2]. \end{aligned}$$

If also $A_3 = 0$,

$$\begin{aligned} x_1 &= A_1^2 D_2^2 . C_3 D_2 (A_1 C_3 - B_1 B_3)^2, \qquad x_2 = A_1^2 D_2^2 . A_1 B_3 (B_3 D_2 - C_2 C_3)^2, \\ x_3 &= - A_1^2 D_2^2 . B_3 C_3 (A_1 C_3 - B_1 B_3)(B_3 D_2 - C_2 C_3). \end{aligned}$$

Using the formula of page 214, we have, for the I_4 of Q,

$$b_1 t^2 (c_3 d_2 t^2 + 4 c_3 e_2 t + 6 d_3 e_2) + d_2 s_4 (6 a_1 b_3 t^2 + 4 a_1 c_3 t + b_1 c_3) = 0,$$

or, more briefly,

$$t^2 \sigma_t + k \rho_t = 0.$$

If $d_3 = 0$, Q has a cusp and the I_4 becomes

$$b_1 c_3 t^3 (d_2 t + 4 e_2) + d_2 s_4 (6 a_1 b_3 t^2 + 4 a_1 c_3 t + b_1 c_3) = 0.$$

If also $b_3 = 0$, Q is bicuspidal and I_4 becomes

$$c_3 [b_1 t^3 (d_2 t + 4 e_2) + d_2 s_4 (4 a_1 t + b_1)] = 0.$$

If $a_1 = d_2 = 1$, $b_1 = 4$ and $e_2 = 1$, so that Q is tricuspidal, I_4 reduces to

$$t^3 (t + 4) + s_4 (t + 1) = 0.$$

In this case the four points in which a line on the meet of the cusp tangents cuts Q form a set of I_4.

It is known that if a conic touch the four bitangents of a rational quartic, the four other common tangents touch the quartic at points which form a set of I_4.* Hence we may infer that *if a conic touch the bitangent and the joins of the three cusps of a tricuspidal quartic, the four other common tangents touch the quartic at points on a line on the meet of the cusp tangents.*

We may verify this inference as follows. For the quartic

$$x_1 = t^4 + 4t^3, \quad x_3 = t + 1, \quad x_3 = t^2,$$

the cusps are at the points

$$(0, 1, 0), \quad (1, 0, 0), \quad (16, 1, -4).$$

Their joins are

$$x_3 = 0, \quad 4x_2 + x_3 = 0, \quad x_1 + 4x_3 = 0.$$

The bitangent is $x_1 + 16x_2 + 12x_3 = 0$, given by $t^2 + 2t + 4 = 0$. These four lines are the base-lines of the range of conics

$$\xi_2(32\xi_1 + \xi_2 - 4\xi_3) + \lambda\xi_1(8\xi_1 + \xi_2 - 2\xi_3) = 0.$$

The line equations of the quartic are

$$\xi_1 = 1, \quad \xi_2 = 2t^3, \quad \xi_3 = -3t(t+2).$$

Substituting, we find for the common tangents

$$2t^3(t^3 + 6t^2 + 12t + 16) + \lambda(t^3 + 3t^2 + 6t + 4) = 0,$$

or, removing the common factor $t^2 + 2t + 4$,

$$2t^3(t+4) + \lambda(t+1) = 0,$$

which is I_4.

Hence, if any parabola be inscribed in the triangle of which the cusps of a deltoid are the vertices, the four common tangents, other than the line at infinity, will touch the deltoid at points which are on a line on its center.

Identifying the osculant cubic of Q with the cubic of page 222, we have

$$\begin{aligned} A_1 &= 4a_1\tau + b_1, & B_1 &= 3b_1\tau, \\ C_2 &= 3d_2, & D_2 &= d_2\tau + 4e_2, \\ A_3 &= b_3, & B_3 &= 3b_3\tau + 2c_3, \\ C_3 &= 2c_3\tau + 3d_3, & D_3 &= d_3\tau, \end{aligned}$$

$$\begin{aligned} R &= 2(6a_1b_3\tau^2 + 4a_1c_3\tau + b_1c_3) \equiv \rho_\tau, \\ S &= -2(c_3d_2\tau^2 + 4c_3e_2\tau + 6d_3e_2) \equiv \sigma_\tau. \end{aligned}$$

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* Stahl: *Crelle*, Vol. CI, p. 302.

Thus we have, for the double point of the osculant,

$$x_1 = 8\,[3\,b_1\tau\,(d_2\tau + 4\,e_2)\,(6\,a_1 b_3\tau^2 + 4\,a_1 c_3\tau + b_1 c_3) \\ + (4\,a_1\tau + b_1)^2\,(c_3 d_2\tau^2 + 4\,c_3 e_2\tau + 6\,d_3 e_2)]^2\,[c_3 d_2\tau^2 + 4\,c_3 e_2\tau + 6\,d_3 e_2],$$

and corresponding expressions for x_2 and x_3.

Varying τ, we have the equations of the nodal locus of osculant cubics. It is a rational curve of the tenth order. The stationary tangent, $x_1 = 0$, of Q cuts it at the points given by $\sigma_\tau = 0$ and touches it at each of the four points given by the other factor in the above expression for x_1; $\sigma_\tau = 0$ gives the two points of Q which form a set of I_4 with $t^2 = 0$; also, when $\sigma_\tau = 0$ the cubic is cuspidal. Since all the osculant cubics touch $x_1 = 0$, this line must be a nodal tangent when a node is on it. The decimic is evidently symmetric as to the six stationary tangents of Q. Hence each stationary tangent cuts it at two points which are cusps of osculants, and touches it at four points at each of which this tangent is a nodal tangent of an osculant. Since the condition that the osculant cubic be cuspidal is of the twelfth degree in τ, this accounts for all the cuspidal osculants. The four points given by the first factor in the expression for x_1 are not, in general, cusps of the decimic, since it can not have twenty-four cusps.

If we let $t = \tau$ in the equation giving the parameters of the node of an osculant, it is of the eighth order in τ, and gives the equation of the eight points of Q at which, in general, the node of the osculant is at the point of contact. It may be easily verified that this equation coincides with that giving the intersections of Q with g_2.

At least eight of the forty intersections of Q with the decimic are at these points. If $d_3 = 0$, Q has a cusp given by $t = 0$. Then $(d_2\tau + 4\,e_2)^2 = 0$ is a common factor of the equations of the decimic, which reduces to an octavic; the osculant cubic at the point given by $d_2\tau + 4e_2 = 0$ degenerates; $d_2 t + 4e_2 = 0$ gives that value of t which with $t^3 = 0$ forms a set of I_4.

When $d_3 = 0$, the osculant at $\tau = \dfrac{-4\,e_2}{d_2}$ is

$$x_1 = -\,[(16\,a_1 e_2 - b_1 d_2)\,t + 12\,b_1 e_2]\,t^2, \quad x_2 = 3\,d_2^2\,t, \\ x_3 = b_3 d_2 t^3 - 2\,(6\,b_3 e_2 - c_3 d_2)\,t^2 - 8\,c_3 e_2 t;$$

or $t = 0$ and

$$x_1 = -\,[(16\,a_1 e_2 - b_1 d_2)\,t + 12\,b_1 e_2]\,t, \quad x_2 = 3\,d_2^2, \\ x_3 = b_3 d_2 t^2 - 2\,(6\,b_3 e_2 - c_3 d_2)\,t - 8\,c_3 e_2.$$

This conic touches $x_2 = 0$, and consequently the three other stationary tangents of Q. If in these equations $t = 0$,

$$x_1 = 0, \quad x_2 = 3\,d_2^2, \quad x_3 = -8\,c_3\,e_2.$$

The point of contact, given by $t = \dfrac{-4\,e_2}{d_2}$, is

$$x_1 = -64\,e_2^2\,(4\,a_1\,e_2 - b_1\,d_2), \quad x_2 = 3\,d_2^4, \quad x_3 = 16\,d_2\,e_2\,(4\,b_3\,e_2 - c_3\,d_2).$$

The join of these points is

$$3\,d_2^3\,(8\,b_3\,e_2 - c_3\,d_2)\,x_1 + 64\,c_3\,e_2^2\,(4\,a_1\,e_2 - b_1\,d_2)\,x_2 \\ + 24\,d_2^2\,e_2\,(4\,a_1\,e_2 - b_1\,d_2)\,x_3 = 0,$$

which proves to be a line of g_2.

We have already shown that, when $d_3 = 0$, $d_2\,t + 4\,e_2 = 0$ gives an intersection of g_2 with Q. Hence, there is only one tangent to g_2 from this point, and consequently there can be only one tangent to the osculant from it.

It is now evident that the conic and any line on the point given by $t = 0$ on it form a curve of the third order which fulfills all the conditions necessary for an osculant cubic.

Returning to the decimic, if also $b_3 = 0$, Q has a second cusp at $t = \infty$. Then $(4\,a_1\,\tau + b_1)^2$ is a common factor of the equations of the octavic which reduces to a sextic; the osculant cubic for this value of τ degenerates; $4\,a_1\,\tau + b_1 = 0$ forms a set of I_4 with $t^3 = \infty$.

The sextic then is

$$x_1 = \tau^3\,(d_2\,\tau + 4\,e_2)\,(2\,a_1\,\tau - b_1)^2, \quad x_2 = (4\,a_1\,\tau + b_1)\,(d_2\,\tau - 2\,e_2)^2, \\ x_3 = 2\,c_3\,\tau^2\,(2\,a_1\,\tau - b_1)\,(d_2\,\tau - 2\,e_2).$$

If $a_1 = d_2 = c_3 = e_2 = 1$ and $b_1 = 4$, this becomes $4\,(\tau - 2)^2 = 0$ and

$$x_1 = \tau^3\,(\tau + 4), \quad x_2 = \tau + 1, \quad x_3 = \tau^2,$$

which now coincides with Q.

The equations of the osculant for this curve are

$$x_1 = 4\,(\tau + 1)\,t^3 + 12\,\tau\,t^2, \quad x_2 = 3\,t + (\tau + 4), \quad x_3 = 2\,t^2 + 2\,\tau\,t.$$

On page 223 we have the equation giving the parameters of the double point of a cubic given in this form. Here

$$\begin{aligned} A_1 &= 4\,(\tau + 1), & B_1 &= 12\,\tau, \\ C_2 &= 3, & D_2 &= \tau + 4, \\ R &= 8\,(\tau + 1), & S &= -2\,\tau\,(\tau + 4). \end{aligned}$$

Hence we find for the parameters of the double point, after removal of the factor

$$64(\tau+1)(\tau+4)(\tau-2),$$

for which values of τ we have seen that the double point is indeterminate,*

$$(t-\tau)(2(\tau+1)t+\tau(\tau+4))=0,$$

thus verifying the statement that on a tricuspidal quartic the node of an osculant cubic is at the point of contact.

We have already (page 211) identified the curve which corresponds to the factor $(\tau-2)^2$ on page 226.

Taking I_4 in the form $t^2\sigma_t+k\rho_t=0$, the two values of k which make this equation self-apolar are given by

$$c_3^2 k(b_1 d_2-4a_1 e_2)+3(d_3 e_2+a_1 b_3 k)^2=0.\dagger$$

Eliminating k, we have for the two self-apolar sets of I_4, after removal of the factor c_3^2,

$$12(a_1 b_3 d_2 t^4+4a_1 b_3 e_2 t^3-4a_1 d_3 e_2 t-b_1 d_3 e_2)^2-(4a_1 e_2-b_1 d_2)t^2\rho_t\sigma_t=0.$$

This equation is the same as that giving the intersections of g_2 with Q.

Hence the two self-apolar sets of I_4 give on Q the intersections of Q with g_2; cubic osculants at these points have double points at the point of contact.

It follows that every set of I_4 is self-apolar when Q has three cusps.

§ 5. *An Interpretation of Syzygies.*

The cubic osculant of

$$x_1=t^4+4t^3,\quad x_2=t+1,\quad x_3=t^2$$

being

$$x_1=4t^3(\tau+1)+12\tau t^2,\quad x_2=3t+\tau+4,\quad x_3=2t^2+2\tau t,$$

the elimination of t gives

$$\begin{aligned}(2x_1x_2^2-6x_2x_3^2-x_3^3)\tau^3-3x_3(x_1x_2+8x_2x_3+x_3^2)\tau^2\\ +3x_3(2x_1x_2+x_1x_3+2x_3^2)\tau-(x_1^2x_2-3x_1x_3^2-8x_3^3)=0.\end{aligned}$$

In lines, the equation of the quartic is

$$\xi_1=1,\quad \xi_2=2t^3,\quad \xi_3=-3t^2-6t.$$

The points of contact of the tangents from the point $(x\xi)=0$ are given by

$$2x_2t^3-3x_3t^2-6x_3t+x_1=0.$$

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* Since $\tau^3(\tau+4)$, $\tau+1$ and $(\tau+2)^3(\tau-2)$ are the sets of I_4 to which the parameters of the cusps belong.

† This equation can not be used if $c_3=0$, since then we can not write the I_4 in the form here used.

The Hessian of this equation is, writing τ for t,

$$H \equiv -\left(x_3(4x_2+x_3)\tau^2 - 2(x_1x_2 - x_3^2)\tau + x_3(x_1+4x_3)\right) = 0.$$

Since the conic osculant of a point cubic touches the stationary tangents, the conic osculant of a line cubic is on the cusps.

The Hessian of a cubic has equal roots when the cubic has two equal roots, and vanishes identically when the cubic has three equal roots.

Therefore H touches the cubic at the point τ and is on the three cusps; *i. e.*, it is the point equation of the conic line osculant. If $\tau = 0$, H is the line osculant at the cusp $(0, 1, 0)$ and degenerates into the joins of this cusp with the other cusps, $(1, 0, 0)$ and $(16, 1, -4)$.*

The coefficient of τ equated to zero gives that conic on the three cusps which touches the cusp tangents $x_1 = 0$ and $x_2 = 0$, given by $\tau = 0$ and $\tau = \infty$.

In general, for the curve $x_i = (\alpha_i t)^n$ *the conic osculant is* $x_i = (\alpha_i t)^2(\alpha_i \tau)^{n-2}$, *and the process of finding the envelope of* $((\alpha_i t)^2(\alpha_i\tau)^{n-2}\xi_i) = 0$ *is the same as that of finding the Hessian of* $((\alpha_i t)^n \xi_i) = 0$.†

The cubic covariant of $2x_2\tau^3 - 3x_3\tau^2 - 6x_3\tau + x_1 = 0$ is

$$\begin{aligned} J \equiv 2\,\big[&(2x_1x_2^2 - 6x_2x_3^2 - x_3^3)\tau^3 - 3x_3(x_1x_2 + 8x_2x_3 + x_3^2)\tau^2 \\ &+ 3x_3(2x_1x_2 + x_1x_3 + 2x_3^2)\tau - (x_1^2x_2 - 3x_1x_3^2 - 8x_3^3)\big] = 0; \end{aligned}$$

i. e., $J = 0$ *is the point cubic osculant of the curve.*

Let U be the cubic $2x_2t^3 - 3x_3t^2 - 6x_3t + x_1$;

D, its discriminant;

H, its Hessian;

J, its cubic covariant;

so that

$U = 0$ is a tangent of the cubic;

$D = 0$ is the point equation;

$H = 0$ is the point equation of the conic line osculant;

$J = 0$ is the point equation of the cubic point osculant.

Then we know there exists a syzygy ‡

$$J^2 = D\,U^2 - 4H^3.$$

---

* Dually the conic osculant of a point cubic at a point of inflexion τ, degenerates into the points where the stationary tangent at τ meets the other stationary tangents.

† Thus for Q, when $t = 0$, $H \equiv 3d_2^2\xi_2^2 + 2(3d_2d_3 - 4c_3e_2)\xi_2\xi_3 + 3d_3^2\xi_3^2 = 0$; *i. e.*, H is the points where $x_1 = 0$ meets K (page 220). H touches g_3 three times, since these points are on g_3 and their join touches g_3 at $(0, 1, 0)$.

‡ Salmon-Fiedler: "Alg. der Lin. Trans.," p. 240.

This says that the points at which U meets J^2 are the points at which it meets H counted three times. But U touches H; therefore U meets J at three coincident points. Hence U is either a stationary tangent, a cusp tangent, or a nodal tangent of J. We can easily show that in some special case it is a nodal tangent; therefore it is so in general. This furnishes another proof of the proposition that the node of the cubic osculant of a tricuspidal quartic is at the point of contact.

The line equation of the conic point osculant, which we shall call H, at the point τ of Q is obtained by forming the Hessian of

$$a_1 \xi_1 \tau^4 + (b_1 \xi_1 + b_3 \xi_3) \tau^3 + c_3 \xi_3 \tau^2 + (d_2 \xi_2 + d_3 \xi_3) \tau + e_2 \xi_2 = 0.$$

The discriminant of the Hessian of a quartic is the product of the discriminant of the quartic and the square of the invariant g_3. Therefore H touches g_3 as well as Q. Dr. Morley has shown that H touches a certain curve three times, which curve must be g_3. H, being also the conic osculant of the cubic osculant at the point τ, touches the stationary tangents of the cubic osculant which are lines of g_3. Hence H touches g_3 at the points where these stationary tangents touch it.*†

We have seen that the cubic osculant of Q at the point τ is a rational point cubic, touching Q at this point, touching the tangents to g_2 from this point and touching the six stationary tangents of Q. These eleven conditions determine the osculant uniquely, although they are not linear.

Let $U = 0$ be the point τ; *i. e.*,

$$U \equiv a_1 \xi_1 \tau^4 + (b_1 \xi_1 + b_3 \xi_3) \tau^3 + c_3 \xi_3 \tau^2 + (d_2 \xi_2 + d_3 \xi_3) \tau + e_2 \xi_2 = 0.$$

The sixteen common lines of the range of quartics $U g_3 + k H g_2 = 0$ are the tangent to H at U counted twice, the two tangents from U to g_2, the six common tangents of g_3 and H, and the six common tangents of g_3 and g_2. The six common tangents of g_3 and H are the three stationary tangents of the cubic osculant at U, each counted twice. These lines, since they are stationary tangents of the osculant, will each count twice as a common tangent of the osculant and any other curve. Hence the cubic osculant belongs to this range when written in line form.

Taking a curve in the form $x_i = (\alpha_i t)^n$, the first osculant is

$$x_i = (\alpha_i \tau)(\alpha_i t)^{n-1},$$

which meets the line $(x \xi) = 0$ when

$$((\alpha_i \tau)(\alpha_i t)^{n-1} \xi_i) = 0.$$

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* *Trans. Am. Math. Soc.*, Vol. VIII, p. 19.

† Each line of g_3 is a stationary tangent of two cubic osculants. Hence two conic osculants touch g_3 at each point and $g_3{}^2$ appears as part of the envelope of H.

Obviously the process of finding the line equation of the osculant is the same as that of finding the Steinerian of $((\alpha_i t)^n \xi_i) = 0$.*

The highest power of τ involved in the line equation of the cubic point osculant of Q obtained in the manner just indicated is τ^4. If we eliminate t from the parametric line equations of the osculant (page 217), τ will occur as high as the sixteenth power in the resultant.

However, σ_τ (and consequently the equation of the twelfth degree in τ giving the parameters of the points of contact of the twelve cuspidal osculants) is obviously a factor of the equation so obtained.

These twelve osculants are curves of the third class.

The sextic covariant of $U=0$ does not give proper osculants of Q. This covariant gives, neglecting the factor 54,

$$\begin{aligned} J \equiv & (8\,a_1^2 \xi_1^2 (d_2 \xi_2 + d_3 \xi_3) - 4\,a_1 c_3 \xi_1 \xi_3 (b_1 \xi_1 + b_3 \xi_3) + (b_1 \xi_1 + b_3 \xi_3)^3)\,\tau^6 \\ & + 2\,(16\,a_1^2 e_2 \xi_1^2 \xi_2 + 2\,a_1 \xi_1 (b_1 \xi_1 + b_3 \xi_3)(d_2 \xi_2 + d_3 \xi_3) - 4\,a_1 c_3^2 \xi_1 \xi_3^2 \\ & \qquad + c_3 (b_1 \xi_1 + b_3 \xi_3)^2 \xi_3)\,\tau^5 \\ & + 5\,(8\,a_1 e_2 \xi_1 \xi_2 (b_1 \xi_1 + b_3 \xi_3) - 4\,a_1 c_3 \xi_1 \xi_3 (d_2 \xi_2 + d_3 \xi_3) \\ & \qquad + (b_1 \xi_1 + b_3 \xi_3)^2 (d_2 \xi_2 + d_3 \xi_3))\,\tau^4 \\ & - 20\,(a_1 \xi_1 (d_2 \xi_2 + d_3 \xi_3)^2 - e_2 \xi_2 (b_1 \xi_1 + b_3 \xi_3)^2)\,\tau^3 \\ & - 5\,(8\,a_1 e_2 \xi_1 \xi_2 (d_2 \xi_2 + d_3 \xi_3) - 4\,c_3 e_2 \xi_2 \xi_3 (b_1 \xi_1 + b_3 \xi_3) \\ & \qquad + (b_1 \xi_1 + b_3 \xi_3)(d_2 \xi_2 + d_3 \xi_3)^2)\,\tau^2 \\ & - 2\,(16\,a_1 e_2^2 \xi_1 \xi_2^2 + 2\,e_2 \xi_2 (b_1 \xi_1 + b_3 \xi_3)(d_2 \xi_2 + d_3 \xi_3) - 4\,c_3^2 e_2 \xi_2 \xi_3^2 \\ & \qquad + c_3 (d_2 \xi_2 + d_3 \xi_3)^2 \xi_3)\,\tau \\ & - (8\,e_2^2 \xi_2^2 (b_1 \xi_1 + b_3 \xi_3) - 4\,c_3 e_2 \xi_2 \xi_3 (d_2 \xi_2 + d_3 \xi_3) + (d_2 \xi_2 + d_3 \xi_3)^3) = 0. \end{aligned}$$

For the quartic $t^2 (a_0 t^2 + 4\,a_1 t + 6\,a_2) = 0$ we find

$$J \equiv -(3\,a_0 a_2 - 2\,a_1^2)(a_1 t + 3\,a_2)\,t^5 = 0.$$

Hence J vanishes identically if U is the product of two squares. If α counts twice as a root of U, α is to be counted five times as a root of J, the sixth root of J being the fourth harmonic of α as to the two other roots of U.

Hence each of the seven† curves obtained by equating to zero the coefficients of τ^6, τ^5, *etc., in* J, *touches the four bitangents of* Q; *also* J *touches* Q *at* U.

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* I take pleasure in thanking Dr. J. R. Conner for noting that the curve of the above range which is the osculant must coincide with the Steinerian of $((a_i t)^n \xi_i) = 0$, and the obvious extension to curves of higher order.

† These seven curves can not be independent; and calculation shows that J is apolar to the sextic giving the parameters of the points of inflexion of Q for all values of ξ_i.

We know that there exists a syzygy,

$$J^2 \equiv g_2 H U^2 - g_3 U^3 - 4H^3.*$$

Hence the six tangents from U to J^2 are the tangents from U to H counted three times. But U is on H, therefore the three tangents from U to J coincide. This means that the tangent to J at U is either a cusp tangent, a stationary tangent or a bitangent. It is easy to show that in some special case it is a bitangent; hence it is so in general.

J and Q have eighteen common tangents. If we substitute in J the values for ξ_i given in the line equations of Q (page 217), we have an equation of the eighteenth order in t and the sixth order in τ.

The octavic in t giving the points of contact of the bitangents of Q is a factor of this, the remaining factor being of the tenth order in t and the sixth order in τ. Of this, $(t-\tau)^5$ must be a factor, which says that the bitangent of J counts five times as a common tangent. Since this line is an ordinary line of Q, J and Q have four coincident common tangents at U.†

The remaining factor is of the fifth order in t and linear in τ. We know that if we ask that the point τ of Q shall be on the tangent to Q at the point t, the equation will contain $(t-\tau)^2$ as a factor. The remaining factor will be of the form

$$(\beta_0 t)^4 \tau^2 + 2(\beta_1 t)^4 \tau + (\beta_2 t)^4 = 0.$$

The fourth harmonic of t as to the roots of this equation in τ will be of the form

$$(\gamma_0 t)^5 \tau + (\gamma_1 t)^5 = 0.$$

Evidently this is the equation which gives the five other common tangents of J and Q.

§ 6. *The Meaning of the Disappearance of Terms.*

If k_i and m_i be respectively the coefficients of t^k and t^m in $(\alpha_i t)^n$, it is evident that if the relation $k_i = \lambda m_i$ exists for one triangle of reference, it will hold for all triangles of reference. If, then, by a change of parameter, we can bring the equations of a curve to a form in which such a relation exists, either the form is one to which the equations of all rational curves of the given order may be reduced, or the relation involves some projective property of the curve.

* Salmon-Fiedler: "Alg. der Lin. Trans.," p. 264.

† Stated dually, Q passes through a node of J, meeting one branch of J in four consecutive points.

Thus the curve

$$x_i = a_i\,(t^n + \lambda\,t^{n-1}) + c_i\,t^{n-2} + \dots + p_i$$

may, by a change of parameter so that $t = \tau - \frac{\lambda}{n}$, be put in the form

$$x_i = a_i\,\tau^n + c_i'\,\tau^{n-2} + d_i'\,\tau^{n-3} + \dots + p_i',$$

which, if we change the reference triangle so that

$$y_1 = |\,x\,a\,c'\,|, \quad y_2 = |\,x\,a\,d'\,|, \quad y_3 = x_3,$$

becomes

$$\begin{aligned} y_1 &= d_1''\,\tau^{n-3} + \dots, \\ y_2 &= c_2''\,\tau^{n-2} + e_2''\,\tau^{n-4} + \dots, \\ y_3 &= a_3\,\tau^n + c_3'\,\tau^{n-2} + d_3'\,\tau^{n-3} + \dots, \end{aligned}$$

whence we see that the curve has a cusp at $\tau = \infty$.

Obviously, if a similar relation had existed between the coefficients of t and the constant terms in $(\alpha_i\,t)^n$, the curve would have had a cusp at $t = 0$, since, if we write $t = \frac{1}{t}$, we merely interchange the coefficients of t^k and t^{n-k}. Conversely, if a curve has a cusp we may write its equations

$$\begin{aligned} x_1 &= d_1\,t^{n-3} + \dots, \\ x_2 &= c_2\,t^{n-2} + \dots, \\ x_3 &= a_3\,t^n + b_3\,t^{n-1} + c_3\,t^{n-2} + \dots, \end{aligned}$$

which, by a change of parameter leaving ∞ unaltered, may be put in a form in which $b_3 = 0$.

In this form one set of the I_n is $t = 0$. Hence it follows that, if a curve has a cusp at $t = \alpha$, one set of the I_n is $(t - \alpha)^{n-1}\,(t - \beta) = 0$. Hence, with each cusp α of a plane rational curve there is associated a definite point, β, of the curve.*

On the other hand, the equations of the general rational cubic may be put in the form

$$x_1 = t^3, \quad x_2 = 1, \quad x_3 = t^2 + \lambda\,t,$$

and for all reference triangles $c_i = \lambda\,b_i$. In this case the relation has no projective meaning.

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* Dually with each stationary tangent, α, there is associated a definite line, β. Thus, for Q we see by inspection that $c_3\,t^6 + 3\,d_3\,t^5 = 0$ is apolar to the expressions for ξ_i in the line equations of the curve (page 217). If, in the equations of the point cubic osculant at the point $t = 0$, we let $t = -\frac{3\,d_3}{c_3}$, we have the coordinates of the tangent to the osculant from the point of contact.

In particular, if $k_i = 0$, then $k_i = \lambda\, m_i$ for all values of m. Evidently the equations of a plane rational curve can not, in general, be reduced to a form in which $k_i = 0$, since this implies that one set of the I_n is of the form $(t-\alpha)^k (t-\beta)^{n-k} = 0$. It is $n-2$ conditions on a binary equation of the n-th order that it be reducible to this form, whereas a set of the I_n is determined by $n-3$ conditions.

Since the necessary and sufficient condition that $t^n = 0$ be apolar to $(\alpha\, t)^n = 0$ is that t be a factor of $(\alpha\, t)^n$, it follows that the necessary and sufficient condition that a plane rational curve of order n degenerate on account of the existence of a common factor in its equations is that a set of the I_n be of the form $(t-\alpha)^n = 0$. That this should be possible is $n-1$ conditions on a binary n-ic, and hence implies two conditions on the coefficients of the curve. Brill, in the article previously cited, using the canonical form

$$x_1 = (\alpha_2\, t)^2 (\alpha_3\, t)^2, \quad x_2 = (\alpha_3\, t)^2 (\alpha_1\, t)^2, \quad x_3 = (\alpha_1\, t)^2 (\alpha_2\, t)^2,$$

notes that if the resultant of any two $(\alpha_i\, t)^2 = 0$ vanishes the quartic degenerates. This is apparently only one condition. However, much care must be used in counting conditions when a canonical form is used. Thus, Salmon's canonical form for the plane cubic

$$x_1^3 + x_2^3 + x_3^3 + 6\, m\, x_1 x_2 x_3 = 0$$

represents three lines if $m = \infty$. This is apparently one condition, but it is of course three conditions on a ternary cubic that it represent three lines.

As defined previously, the m-th osculant of a curve is

$$x_i = (\tau D_t + \tau_1 D_{t_1})^m (a_i t^n + \ldots + k_i t^k t_1^{n-k} + \ldots + p_i t_1^n).$$

The equations of the $(n-k)$-th osculant at the point $\dfrac{\tau}{\tau_1} = 0$ will, in general, contain terms involving t to the k-th power; but, if $k_i = 0$, t^k disappears and the osculant degenerates. In like manner the equations of the k-th osculant at the point $\dfrac{\tau}{\tau_1} = \infty$ will, in general, contain terms not involving t; but, if $k_i = 0$, these terms disappear: t is a factor of the three x_i and the curve again degenerates. Hence, if a set of the I_n is of the form $(t-\alpha)^k (t-\beta)^{n-k} = 0$, the k-th osculant at the point α, and the $(n-k)$-th osculant at the point β, degenerate.

If $k = n-1$, this means that the $(n-1)$-st osculant (*i.e.*, the tangent) at the cusp, and the first osculant at the associated point β, degenerate.

Thus, the third osculant of

$$x_1 = t^4 + 4\,t^3, \quad x_2 = t + 1, \quad x_3 = t^2$$

is

$$x_1 = 4\,\tau^2(\tau + 3)\,t + 4\,\tau^3, \quad x_2 = t + 3\,\tau + 4, \quad x_3 = 2\,\tau\,t + 2\,\tau^2.$$

If $t = -2$, which gives a cusp of the quartic, the osculant becomes

$$x_1 = 16\,(t - 2), \quad x_2 = t - 2, \quad x_3 = -4\,(t - 2).$$

It has been shown before that the first osculant at the associated point $t = 2$ degenerates (page 208). We have seen (page 211) that, if we give a bicuspidal quartic a third cusp, the cubic osculant at the point associated with this cusp degenerates, and that the conic part of it is a part of the locus of nodes of osculant cubics. We have now shown that on any plane rational cuspidal curve the $(n-1)$-st osculant at the point associated with the cusp degenerates. From the symmetrical manner in which the equations of osculants are formed, we may infer that, in general, that curve of the $(n-2)$-nd order, which is part of the degenerate $(n-1)$-st osculant at the point associated with the cusp, is also a part of the locus of nodes of $(n-1)$-st osculants.

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